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Dynamic Programming Approach to Optimal Feedback-Feedforward Controller Design

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SUMMARY

A new derivation of the parametric expansion of the dynamic programming algorithm for optimal feedback-feedforward controller design is presented. Matrix methods are employed to give clarity to the results first offered by Merriam. The advantages of the optimal design technique for multivariable problems are readily apparent. A unique technique is given for easily solving the design equations for the case of time invariant systems with gaussian load disturbance.

1. Introduction

Optimal feedback controllers have been studied by many people over the last few years. The literature abounds with a large variety of systems used in conjunction with recently developed optimal control theory [1]. Feedforward or "invariant" controllers have enjoyed a high degree of popularity, especially in the chemical industry [8, 3, 4]. Luecke and McGuire [7] obtained an optimum composite feedforward-feedback controller using Weiner's frequency domain methods [10]. While they observed good results, the Weiner method suffers from restrictions which exclude multivariable systems from consideration. The control design method described herein does not include any restriction on time-varying multivariable systems.

2. Derivation of Design Equations

Consider a linear dynamic system of the form

$$\dot{x}(t) = Bx(t) + Cm(t) + Du(t) \qquad (1)$$
$$q(t) = Ax(t)$$

where

$x(t)$ = state vector
$m(t)$ = control vector
$u(t)$ = disturbance vector
$q(t)$ = output vector
$A, B, C, D = n\times n$ coefficient matrices.

A large number of random load disturbance signals can be characterized by the following autocorrelation function

$$\Theta_{uu} = \bar{u}^2 \, e^{-\nu|\tau|}; \qquad \tau \in \langle -\infty, \infty \rangle \qquad (2)$$

where $\bar{u}$ = mean square load disturbance
ν = disturbance frequency in radians per unit time.

The familiar gaussian signal representation [6] will be used.

Define a scalar performance index

$$e(t) = \int_t^T \{\langle q(\mu), \Phi q(\mu)\rangle + \langle m(\mu), \Psi m(\mu)\rangle\} \, d\mu \qquad (3)$$

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where T = terminal time boundary
Φ = non-negative definite diagonal output weighting matrix
Ψ = positive definite diagonal control weighting matrix

By imposing the restriction of non-negative and positive definite weighting factors, the extremal of $e(t)$ must exist.

Since the process is subject to random load disturbances and measurement noise, the integrand is dependent on a particular ensemble. The conditional mean can be used to rid the performance index—hence the optimal controller—of its ensemble dependence.

$$\overline{e(t)}^t = \int_{-\infty}^{\infty} e\,p(e|y;t)\,de \tag{4}$$

where e = amplitude of $e(t)$
y = amplitude of measured signals at time t
p = conditional probability density

The conditional probability density is the probability of amplitude e at time t given the measured values of y at the same time t. Then the performance index to be extremized for each fixed value of time t is

$$\overline{e(t)}^t = \int_t^T \{\overline{\langle q(\mu), \Phi q(\mu)\rangle}^t + \overline{\langle m(\mu), \Psi m(\mu)\rangle}^t\}\,d\mu\,. \tag{5}$$

In order to use the dynamic programming procedures, a minimum performance index must be defined.

$$E(\overline{x(\sigma)}^t, \sigma) = \underset{m(\sigma)\in M}{\text{Min}}\ \overline{e(\sigma)}^t \tag{6}$$

where M = closed set of allowable control signal values.

The continuous form of the dynamic programming algorithm [9] is

$$\underset{m(\sigma)}{\text{Min}}\left\{\overline{q(\sigma), \Phi q(\sigma)\rangle}^t + \overline{\langle m(\sigma), \Psi m(\sigma)\rangle}^t + \left(\frac{\partial E[\overline{x(\sigma)}^t, \sigma]}{\partial \overline{x(\sigma)}^t}\right)^T \overline{\dot{x}(\sigma)}^t\right\} = -\frac{\partial E[\overline{x(\sigma)}^t, \sigma]}{\partial \sigma} \tag{7}$$

For linear systems it has been shown [12] that the minimum performance index is at most a quadratic function of the state variable. Therefore the minimum performance index can be expanded in a truncated Taylor series in terms of three unknown parameters.

$$E[\overline{x(\sigma)}^t, \sigma] = I(\sigma) - 2(\overline{x(\sigma)}^t)^T J(\sigma) + (\overline{x(\sigma)}^t)^T K(\sigma)\overline{x(\sigma)}^t$$

where $I(\sigma)$ is a scalar element, $J(\sigma)$ is a n-element feedforward vector, and $K(\sigma)$ is a symmetric $n\times n$ feedback matrix.

The partial derivatives of $E(\overline{x(\sigma)}^t, \sigma)$ needed to evaluate equation (7) are

$$\left(\frac{\partial E[\overline{x(\sigma)}^t, \sigma]}{\partial \overline{x(\sigma)}^t}\right)^T = -2J^T(\sigma) + 2(\overline{x(\sigma)}^t)^T K(\sigma) \tag{9}$$

$$\frac{\partial E[\overline{x(\sigma)}^t, \sigma]}{\partial \sigma} = \dot{I}(\sigma) - 2(\overline{x(\sigma)}^t)^T \dot{J}(\sigma) + (\overline{x(\sigma)}^t)^T \dot{K}(\sigma)\overline{x(\sigma)}^t \tag{10}$$

Carrying out the minimization procedure indicated in equation (7), the optimal control equation is

$$m^*(t) = \Psi^{-1}C^T J(t) - \Psi^{-1}C^T K(t)\overline{x(t)}^t \tag{11}$$

where $m^* \in M$

To simplify notation the conditional mean and time arguments are deleted from the rest of this discussion.

Substituting equations (9), (10), and (11) into equation (7) results in

$$-\dot{I}+2x^T\dot{J}-x^T\dot{K}x = x^TA^T\Phi Ax+(-2J^T+2x^TK)Du -\tfrac{1}{4}(-2J^T+2x^TK)C\Psi^{-1}C^T(-2J)+2Kx +(-2J^T+2x^TK)Bx. \quad (12)$$

Expanding this equation and collecting terms with like powers of the state variable, x, gives

$$-\dot{I}+2x^T\dot{J}-x^TKx = \{J^TC\Psi^{-1}C^TJ-2J^TDu\} +x^T\{+2KC\Psi^{-1}C^TJ-2B^TJ+2KDu\} +x^T\{A^T\Phi A-KC\Psi^{-1}C^TK+2KB\}x \quad (13)$$

This equation must hold for all possible values of the state variable x, so like powers of x on each side of the equation can be equated.

In order to be able to equate the quadratic terms, the factors within the braces of the right-hand side quadratic must be shown to be symmetric because the left-hand side, K, is symmetric. The first two factors are already symmetric, so it remains to be shown that x^T2KBx can be manipulated into an equivalent symmetric form

$$2KB = (KB+B^TK)+(KB-B^TK). \quad (14)$$

The first term of the right-hand side of this equation is symmetric while the second term is skew-symmetric. It can be shown that the quadratic form of the skew-symmetric matrix is zero.

Consider two arbitrary vectors y and z which are related by

$$y = Bz \quad (15)$$

Then

$$z^T(KB-B^TK)z = z^TKBz-z^TB^TKz \quad (16)$$

or

$$z^T(KB-B^TK)z = z^TKy-y^TKz = 0 \quad (17)$$

because K is symmetric. Therefore the symmetry of the quadratic term has been demonstrated.

Then the equations which must be solved to determine the unknown parameters are

$$\dot{I}(\sigma) = J^T(\sigma)Du-J^T(\sigma)C\Psi^{-1}C^TJ(\sigma) \quad (18)$$

$$\dot{K}(\sigma) = K(\sigma)C\Psi^{-1}C^TK(\sigma)-B^TK(\sigma)-K(\sigma)B-A^T\Phi A \quad (19)$$

$$\dot{J}(\sigma) = K(\sigma)C\Psi^{-1}C^TJ(\sigma)-B^TJ(\sigma)+KD\overline{u(\sigma)}^t. \quad (20)$$

The boundary conditions for this equation follow directly from equation (3)

$$I(T) = J(T) = K(T) = 0. \quad (21)$$

Merriam [9] arrived at the same equations by way of a complicated summation notation operation. Because of typographical errors and the unfortunate complexity of notation, his results have not received wide attention.

3. Separable Load Disturbances

Equation (18) is not convenient for control purposes in its present form. Because the load disturbance was restricted to gaussian statistics, a further simplification results. It is well known [11] that the conditional mean of a gaussian signal can be written in a separable form

$$\overline{u(\sigma)}^t = U(\sigma, t)\overline{u(t)}^t \quad (22)$$

where $U(\sigma, t)$ = ratio of the autocorrelation functions of u at time t and time σ.
Notice that $\overline{u(t)}^t$ is the measurable disturbance signal.

A new parameter can be defined

$$J(t) = S(t)\overline{u(t)}^t . \tag{23}$$

Substituting this expression into equation (19) and rearranging results gives

$$\dot{S}(\sigma) = [K(\sigma)C\Psi^{-1}C^T - B^T]S(\sigma) + K(\sigma)DU(\sigma, t) . \tag{24}$$

In terms of this new parameter, the optimal control equation is

$$m^*(t) = \{\Psi^{-1}C^T S(t)\}\overline{u(t)}^t + \{\Psi^{-1}C^T K(t)\}\overline{x(t)}^t . \tag{25}$$

The important feature of this new control equation is that the control signal is now a direct function of the two measurable signals of the process $\overline{u(t)}^t$ and $\overline{x(t)}^t$. Therefore, the terms within brackets in equation (25) are the feedforward and feedback controller gains.

Once the system dynamics {equation (1)} and the autocorrelation function of the load disturbance {equation (2)} are known, and the constraints {equation (3)} are chosen, equations (20) and (24) must be solved. Then equation (25) furnishes the configuration and the various gains required for an optimal composite feedforward-feedback control system.

4. Time-Invariant Regulatory Control

For the practical case of designing a regulatory control system ($T \to \infty$), these equations are readily solved. Equation (20) is the matrix riccatti differential equation which has received considerable attention in recent years. There are several good methods [5] available for solving this equation to obtain the feedback parameter, K.

The impulse response solution of equation (24) is

$$S(t) = \varphi(t-T)S(T) + \int_T^t \varphi(t-\alpha)KDU(\alpha-t)d\alpha \tag{26}$$

where $\varphi(t)$ is the fundamental matrix or matrizant of the homogeneous part of equation (24). Implementing the boundary condition and rearranging

$$S(t) = -\int_0^{T-t} \varphi(-\varepsilon)KDU(\varepsilon)d\varepsilon . \tag{27}$$

Since we are considering the time-invariant ($T \to \infty$) problem, this equation becomes

$$S = -\int_0^{\infty} \varphi(-\varepsilon)KDU(\varepsilon)d\varepsilon . \tag{28}$$

For gaussian signals

$$U(\varepsilon) = \frac{\Theta(\varepsilon)}{\Theta(0)} \quad \text{for} \quad \varepsilon > 0 . \tag{29}$$

Then from equation (2) the separable portion of the load disturbance is

$$U(\varepsilon) = e^{\bar{\alpha}\varepsilon} \tag{30}$$

$$\text{where } \bar{\alpha} = \begin{bmatrix} \alpha & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{bmatrix}$$

The diagonal elements of the matrix exponential contain the frequencies of the various input disturbances. The off-diagonal elements would normally be zero for the vast majority of practical problems.

Equation (23) becomes

$$S = -\int_0^\infty e^{P\varepsilon} KD\, e^{\bar{\alpha}\varepsilon} d\varepsilon \tag{31}$$

where $P = [B^T - KC\Psi^{-1}C^T K]$.

Bellman [2] has shown that the above equation is the solution of the matrix equation

$$PS + S\bar{\alpha} = KD \tag{32}$$

provided the integral exists for all KD. Furthermore, the necessary and sufficient conditions that this integral exists is that $\mu_i + \lambda_j \leqq 0$, where μ_i, λ_j are the characteristic roots of P and $\bar{\alpha}$ respectively [2]. For physically realizable systems, the above conditions are always met because the real parts of the characteristic roots of these matrices must be negative for stable systems.

Rearranging equation (32) into a form convenient for trial and error solution gives

$$S^{j+1} = P^{-1}\bar{\alpha} S^j \tag{33}$$

where j = the iteration index.

The feedforward gain, S, on the right-hand side of this equation is used to calculate a new matrix S^{j+1}. This procedure would be continued until some convergence criteria is satisfied.

Actually equation (32) may be solved directly.

$$P \times I + I \times a^{-T} S = KD \tag{34}$$

or

$$S = P \times I + I \times a^{-T-1} KD\,. \tag{35}$$

The symbol, $\times$, denotes the first power Kronecker product and is defined as

$$A \times B = (a_{ij}B) \qquad i, j = 1, \ldots, n \tag{36}$$

Note that the Kronecker product of two n-dimensional matrices is an n^2-dimensional matrix. This quadratic increase in dimensionality unfortunately limits the practicality of this solution method, so the iteration scheme is the preferred method of solution.

5. An Example

The application of the previous mathematical development to the optimal control of a simple process emphasizes the power of the parametric expansion technique. A perfectly mixed continuous reactor can be represented by the following first order dynamics

$$\dot{x} = -ax + bm + cu \tag{37}$$

where x = system output
m = control variable
u = disturbance
a, b, c = system gains.

The optimal regulatory control of this process can be described by a scalar performance index

$$e(t) = \int_t^\infty \{x^2 + m^2\}\, dt\,. \tag{38}$$

Since gaussian characteristics of the stochastic disturbances are common in the chemical industry, the autocorrelation of the variable, u, can be described as follows:

$$\Theta_{uu}(T) = u^2 e^{-\nu|T|} \tag{39}$$

where u^2 = mean square amplitude
ν = frequency

The optimal control equation follows directly from equation (25).

$$m^*(t) = \frac{cS}{\Psi} u(t) + \frac{cK}{\Psi} x(t) \tag{40}$$

where the parameter, S, is defined by equation (28) and the parameter, K, is defined by equation (19).

An analytical expression for the optimal control equation is readily obtained by solving for S and K.

$$m^*(t) = Q_{FF} u(t) - Q_{FB} x(t) \tag{41}$$

$$Q_{FF} = -\frac{c}{b} \frac{-a + (a^2 + b^2/\Psi)^{\frac{1}{2}}}{v + (a^2 + b^2/\Psi)^{\frac{1}{2}}} \tag{42}$$

$$Q_{FB} = \frac{-a + (a^2 + b^2/\Psi)^{\frac{1}{2}}}{b} \tag{43}$$

The parameter, Q_{FF}, is the feedforward controller gain, while the parameter, Q_{FB}, is the familiar feedback gain.

A very important point to be emphasized here is that the controller is a composite of both feedforward and feedback modes. Most control designers neglect the disturbances thereby getting only the feedback mode.

Perfect control is achieved if the penalty on the control action approaches infinity ($\Psi \to \infty$). A brief examination of the control parameters discloses that ideal feedforward and infinite feedback are specifics.

$$\lim_{\Psi \to \infty} Q_{FF} = -\frac{c}{b} \tag{44}$$

$$\lim_{\Psi \to \infty} Q_{FB} = \infty \tag{45}$$

These results are in agreement with previous results. The difference in this approach is that the work of control is divided by the two different types of control operational modes.

6. Conclusions

The technique described herein is practical and easily implemented for the design of optimal regulatory control systems. More complicated time-varying systems could also be attacked with this method, but considerably more calculations (such as obtaining the matrix exponential) would be required. This design technique is more general than the previously used frequency domain methods.

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